

MONODROMY FOR SYSTEMS OF VECTOR BUNDLES AND MULTIPLICATIVE PREPROJECTIVE ALGEBRAS

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ABSTRACT. We study systems involving vector bundles and logarithmic connections on Riemann surfaces and linear algebra data linking their residues. This generalizes representations of deformed preprojective algebras. Our main result is the existence of a monodromy functor from such systems to representations of a multiplicative preprojective algebra. As a corollary, we prove that the multiplicative preprojective algebra associated to a Dynkin quiver is isomorphic to the usual preprojective algebra.

1. INTRODUCTION

In this paper we consider systems of vector bundles and logarithmic connections on Riemann surfaces, combined with a certain linking between them which involves the residues of the connections. The configuration is described by specifying both a quiver and a Riemann surface with finitely many connected components.

Our main result is the existence of a monodromy functor which, in case the Riemann surface is compact, gives an equivalence between a category of such systems and a category of representations of a multiplicative preprojective algebra, in the sense of [6].

In previous work (see [4] for a survey) we used deformed preprojective algebras, multiplicative preprojective algebras and monodromy to study questions about the existence of tuples of square matrices in prescribed conjugacy classes whose sum is zero or whose product is the identity. In that work the quiver was taken to be star-shaped, and the restriction of a representation to each arm of the star fixed one of the matrices. This paper generalizes some of that work, and provides a more natural setting, as it enables one apply monodromy in a quiver setting, without having to pass to and from tuples of matrices.

As an application, we prove that the multiplicative preprojective algebra $\Lambda^1(Q)$ associated to a Dynkin quiver is isomorphic to the usual

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preprojective algebra $\Pi(Q)$, answering a question of Shaw [9]. We also give an interpretation of this as a special case of Hilbert's 21st problem.

The structure of the paper is as follows. In section 2 we introduce the notion of a 'Riemann surface quiver', which combines a quiver with a Riemann surface, and we also introduce 'vector bundle representations'. In section 3 we introduce the idea of a ' λ -connection system' on a vector bundle representation, combining a logarithmic connection with linear algebra data linking its residues. In section 4 we prove a 'Lifting Theorem', which gives a criterion for the existence of λ -connection systems, and in section 5 we discuss the 'Monodromy Theorem', giving the existence of a monodromy functor; the proof uses two lemmas, which are given in sections 6 and 7. In section 8 we identify the target of the monodromy functor in the compact case with a category of representations of a multiplicative preprojective algebra. Finally, we address Shaw's question and Hilbert's 21st problem in section 9.

I would like to thank R. Bielawski and T. Hausel for suggesting that the constructions in this paper might be considered in the context of torsion-free sheaves on a singular curve. I hope to return to this in another paper.

2. VECTOR BUNDLE REPRESENTATIONS

By a *Riemann surface quiver* Γ we mean a quiver whose set of vertices has the structure of a Riemann surface X , not necessarily connected, but with only finitely many connected components. We assume in addition that Γ has only finitely many arrows. We say that Γ is *compact* if X is compact, and that Γ is of \mathbb{P}^1 -*type* if each connected component is a copy of the Riemann sphere \mathbb{P}^1 .

By a *marked* point, we mean a point of the Riemann surface which occurs as a head or tail of an arrow. We say that Γ has *non-interfering arrows* provided that each point of the Riemann surface occurs at most once as the head or tail of an arrow, so that if there are n arrows, then there are exactly $2n$ marked points.

Let X_i ($i \in I$) be the connected components of X , for a suitable finite indexing set I . Given $p \in X$ we denote by $[p]$ the index of the connected component containing p . We define the *component quiver* $[\Gamma]$ of Γ to be the quiver with vertex set I and with an arrow $[a] : [p] \rightarrow [q]$ for each arrow $a : p \rightarrow q$ in Γ .

By a *vector bundle representation* of a Riemann surface quiver Γ we mean a representation of the quiver such that the vector spaces at each vertex form the fibres of a (holomorphic) vector bundle. Equivalently it is a collection $\mathbf{E} = (E, E_a)$ consisting of a vector bundle E over X

and linear maps $E_a : E_p \rightarrow E_q$ for each arrow $a : p \rightarrow q$ in Γ . Here E_p denotes the fibre of E at p . Note that specifying the vector bundle E over X is equivalent to specifying a vector bundle E_i on each X_i , and so we can also consider a vector bundle system as a collection (E_i, E_a) . We do not require the E_i to all have the same rank, and define the *dimension vector* of \mathbf{E} to be the vector in \mathbb{N}^I whose i th component is $\text{rank } E_i$. If Γ is compact, the *degree* of \mathbf{E} is $\deg E = \sum_{i \in I} \deg E_i$.

There is a natural category \mathcal{V}_Γ of vector bundle representations of Γ , in which a homomorphism θ from $\mathbf{E} = (E, E_a)$ to $\mathbf{E}' = (E', E'_a)$ is a vector bundle homomorphism $\theta : E \rightarrow E'$ (or equivalently a collection of vector bundle homomorphisms $\theta_i : E_i \rightarrow E'_i$) with the property that $\theta_q E_a = E'_a \theta_p$ for each arrow $a : p \rightarrow q$. This is clearly an additive category over \mathbb{C} with split idempotents. If Γ is compact it has finite-dimensional homomorphism spaces.

We say that a vector bundle representation \mathbf{E} is *vb-trivial* if the bundles E_i are trivial vector bundles (of some rank) for all i . If Γ is compact then the category of trivial vector bundles on X_i is equivalent to the category of vector spaces, and hence the category of vb-trivial vector bundle representations of Γ is equivalent to the category of representations of the component quiver.

3. CONNECTIONS

Henceforth we fix a Riemann surface quiver Γ with underlying Riemann surface X . Let D be the set of marked points of X . We fix a collection of scalars $\lambda = (\lambda_p)_{p \in D} \in \mathbb{C}^D$. Let $D_i = D \cap X_i$, and define $\lambda_i = \sum_{p \in D_i} \lambda_p$ for $i \in I$.

By a λ -*connection system* on a vector bundle representation \mathbf{E} of Γ , we mean a collection $\nabla = (\nabla, \nabla_a)$ consisting of a connection $\nabla : E \rightarrow \Omega_X^1(\log D) \otimes E$ on E , holomorphic except possibly for logarithmic poles on D (or equivalently a connection ∇_i on each E_i), and linear maps $\nabla_a : E_q \rightarrow E_p$ for each arrow $a : p \rightarrow q$ in Γ , satisfying

$$\text{Res}_p \nabla - \lambda_p 1_{E_p} = \sum_{t(a)=p} \nabla_a E_a - \sum_{h(a)=p} E_a \nabla_a$$

for points $p \in D$. If the Riemann surface quiver has non-interfering arrows we can rewrite this as

$$\text{Res}_p \nabla - \lambda_p 1_{E_p} = \nabla_a E_a \quad \text{and} \quad \text{Res}_q \nabla - \lambda_q 1_{E_q} = -E_a \nabla_a$$

for each arrow $a : p \rightarrow q$ in Γ .

We denote by $\mathcal{C}_{\Gamma, \lambda}$ the category whose objects are pairs (\mathbf{E}, ∇) , consisting of a vector bundle representation \mathbf{E} of Γ and a λ -connection system ∇ on \mathbf{E} , and in which a morphism from (\mathbf{E}, ∇) to (\mathbf{E}', ∇') is

by definition a morphism of vector bundle representations $\theta : \mathbf{E} \rightarrow \mathbf{E}'$ with the property that the square

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & \Omega_X^1(\log D) \otimes E \\ \theta \downarrow & & \downarrow 1 \otimes \theta \\ E' & \xrightarrow{\nabla'} & \Omega_X^1(\log D) \otimes E' \end{array}$$

commutes and $\theta_p \nabla_a = \nabla'_a \theta_q$ for each arrow $a : p \rightarrow q$ in Γ . Note the following interpretation in terms of deformed preprojective algebras [5].

Proposition 1. *If Γ is of \mathbb{P}^1 -type, then the full subcategory of $\mathcal{C}_{\Gamma, \lambda}$ given by pairs (\mathbf{E}, ∇) with \mathbf{E} vb-trivial is equivalent to the category of representations of $\Pi^\lambda([\Gamma])$, the deformed preprojective algebra associated to the component quiver $[\Gamma]$ of Γ .*

Proof. As the E_i are trivial vector bundles, all fibres E_p ($p \in X_i$) can all be identified, say with vector space V_i , and then these spaces together with the linear maps E_a and ∇_a define a representation of the double quiver of $[\Gamma]$.

Any logarithmic connection on E_i has the form

$$\nabla_i(f) = df + A_i(z) \cdot f dz$$

for f is a section of E_i , where $A_i(z)$ is an $\text{End}(V_i)$ -valued function of z , holomorphic except for simple poles on D_i . If z is a coordinate for X_i with $\infty \notin D_i$, then $A_i(z)$ must have the form

$$A_i(z) = \sum_{p \in D_i} \frac{R_p}{z - p}$$

where the residues $R_p = \text{Res}_p \nabla_i \in \text{End}(V_i)$ are any endomorphisms, subject to the relation $\sum_{p \in D_i} R_p = 0$; see for example [1, §1.2]. Using the linking of residues, this relation can be rewritten as

$$\sum_{h(a) \in X_i} E_a \nabla_a - \sum_{t(a) \in X_i} \nabla_a E_a = \lambda_i 1_{V_i}.$$

This is the relation at vertex i for the deformed preprojective algebra. The assertion follows. \square

4. LIFTING THEOREM

The following is a straightforward analogue of [2, Theorem 3.3] and [3, Theorem 7.1].

Theorem 1. *If \mathbf{E} is a vector bundle representation of a compact Riemann surface quiver Γ , then there exists a λ -connection system on \mathbf{E} if and only if*

$$\deg E' + \sum_{i \in I} \lambda_i \operatorname{rank} E'_i = 0$$

for every direct summand \mathbf{E}' of \mathbf{E} .

Proof. We identify $f \in \operatorname{End}(E)$ with $(f_i) \in \bigoplus_{i \in I} \operatorname{End}(E_i)$. Since X_i is compact, there is a trace mapping $\operatorname{tr}_i : \operatorname{End}(E) \rightarrow \mathbb{C}$ with $\operatorname{tr}(f) = \operatorname{tr}(f_p)$ for all $p \in X_i$.

A criterion of Mihai [8] for the existence of logarithmic connections, see [3, Theorem 7.2], implies that there is a λ -connection system on \mathbf{E} if and only if there are linear maps $\nabla_a : E_q \rightarrow E_p$ for all arrows $a : p \rightarrow q$ in Γ satisfying

$$\sum_{p \in D} \operatorname{tr} \left((\lambda_p 1 + \sum_{t(a)=p} \nabla_a E_a - \sum_{h(a)=p} E_a \nabla_a) f_p \right) = \frac{1}{2\pi\sqrt{-1}} \langle b(E), f \rangle$$

for all $f \in \operatorname{End}(E)$, where $\langle b(E), f \rangle = \sum_{i \in I} \langle b(E_i), f_i \rangle$.

There is an exact sequence

$$0 \rightarrow \operatorname{End}(\mathbf{E}) \rightarrow \operatorname{End}(E) \rightarrow \bigoplus_{a:p \rightarrow q} \operatorname{Hom}(E_p, E_q)$$

where the second map sends f to the tuple whose a th component is $E_a f_p - f_q E_a$. Using the trace pairing it dualizes to give an exact sequence

$$\bigoplus_{a:p \rightarrow q} \operatorname{Hom}(E_q, E_p) \xrightarrow{F} \operatorname{End}(E)^* \xrightarrow{G} \operatorname{End}(\mathbf{E})^* \rightarrow 0$$

where F sends a tuple (∇_a) to the linear form sending f to

$$\sum_{a:p \rightarrow q} \operatorname{tr} \left((E_a f_p - f_q E_a) \nabla_a \right) = \sum_{p \in D} \operatorname{tr} \left(\left(\sum_{t(a)=p} \nabla_a E_a - \sum_{h(a)=p} E_a \nabla_a \right) f_p \right).$$

If we define $\xi \in \operatorname{End}(E)^*$ by

$$\xi(f) = \frac{1}{2\pi\sqrt{-1}} \langle b(E), f \rangle - \sum_{i \in I} \sum_{p \in D_i} \lambda_i \operatorname{tr}_i(f)$$

then the condition for there to be a λ -connection system can be written as $\xi \in \operatorname{Im}(F)$, so as $\xi \in \operatorname{Ker}(G)$, so as $\xi(f) = 0$ for all $f \in \operatorname{End}(\mathbf{E})$. Now if $f = p_{\mathbf{E}'}$, the projection onto a direct summand \mathbf{E}' of \mathbf{E} , then

$$\xi(p_{\mathbf{E}'}) = \sum_{i \in I} \left(-\deg E'_i - \sum_{p \in X_i} \lambda_i \operatorname{rank} E'_i \right),$$

so the existence of a λ -connection system implies the condition in the statement of the Theorem. Conversely, since $\xi(f) = 0$ for f nilpotent, the stated condition implies the existence of a λ -connection system on any indecomposable direct summand of \mathbf{E} , and by combining these one obtains a λ -connection system on \mathbf{E} . \square

5. MONODROMY THEOREM

Let Γ be a Riemann surface quiver with non-interfering arrows, and let $\lambda \in \mathbb{C}^D$, as before. We fix a subset T of \mathbb{C} . We assume that T is *non-resonant*, by which we mean that distinct elements of T never differ by an integer, and we assume moreover that $0 \in T$.

We say that a λ -connection system ∇ *has eigenvalues in T* provided that $E_a \nabla_a$ has eigenvalues in T for all arrows a in Γ . (Since $0 \in T$, it is equivalent that $\nabla_a E_a$ has eigenvalues in T .) We denote by $\mathcal{C}_{\Gamma, \lambda}^T$ the full subcategory of $\mathcal{C}_{\Gamma, \lambda}$ given by the pairs (\mathbf{E}, ∇) such that ∇ has eigenvalues in T .

For each $i \in I$, fix a base point b_i for $X_i \setminus D_i$, and for each $p \in D_i$ fix a loop $\ell_p \in \pi_1(X_i \setminus D_i, b_i)$ around p . Define $\sigma \in (\mathbb{C}^\times)^D$ by $\sigma_p = e^{2\pi\sqrt{-1}\lambda_p}$ for $p \in D$.

We denote by $\text{Rep}_\sigma \pi(\Gamma)$ the category whose objects are given by collections $(V_i, \rho_i, \rho_a, \rho_a^*)$ consisting of a representation $\rho_i(\pi_1(X_i \setminus D_i, b_i)) \rightarrow \text{GL}(V_i)$ for each $i \in I$, and linear maps $\rho_a : V_i \rightarrow V_j$ and $\rho_a^* : V_j \rightarrow V_i$ for each arrow $a : p \rightarrow q$ in Γ , where $i = [p]$ and $j = [q]$, satisfying

$$\sigma_p^{-1} \rho_i(\ell_p)^{-1} = 1_{V_i} + \rho_a^* \rho_a \quad \text{and} \quad \sigma_q \rho_j(\ell_q) = 1_{V_j} + \rho_a \rho_a^*$$

and whose morphisms are the natural ones.

Let $S = \{e^{2\pi\sqrt{-1}t} - 1 : t \in T\}$. We denote by $\text{Rep}_\sigma^S \pi(\Gamma)$ the full subcategory of $\text{Rep}_\sigma \pi(\Gamma)$ consisting of the collections in which $\rho_a \rho_a^*$ has eigenvalues in S for all arrows a . (Since $0 \in S$, it is equivalent that $\rho_a^* \rho_a$ has eigenvalues in S .)

Theorem 2. *There is an equivalence of categories $\mathcal{C}_{\Gamma, \lambda}^T \rightarrow \text{Rep}_\sigma^S \pi(\Gamma)$.*

Note the following special cases: if $T = \{0\}$, the theorem gives an equivalence $\mathcal{C}_{\Gamma, \lambda}^{\{0\}} \rightarrow \text{Rep}_\sigma^{\{0\}} \pi(\Gamma)$; if T is a transversal to \mathbb{Z} in \mathbb{C} such as $\{z \in \mathbb{C} : 0 \leq \Re z < 1\}$, it gives an equivalence $\mathcal{C}_{\Gamma, \lambda}^T \rightarrow \text{Rep}_\sigma \pi(\Gamma)$. Note also that one could specify a different set T for each arrow a in Γ .

Proof. Deleting all arrows and marked points, monodromy gives an equivalence from the category consisting of vector bundles on $X \setminus D$ equipped with a holomorphic connection, to the category whose objects are a collection of representations (ρ_i) of the fundamental groups $\pi_1(X_i \setminus D_i, b_i)$, see for example [7, Theorem 1.1].

We need to check that the ways of extending the vector bundles and connections to give vector bundle representations and λ -connection systems on Γ correspond to the ways of extending the representations of the fundamental groups to give objects in $\text{Rep}_\sigma^S \pi(\Gamma)$.

In fact it suffices to check this locally for each arrow $a : p \rightarrow q$ in Γ . Thus, replacing X by the union of sufficiently small disk neighbourhoods of p and q , we may suppose that X is the disconnected union of two disks $X_1 \cup X_2$, that p and q are the centres of these disks, and that $a : p \rightarrow q$ is the only arrow in Γ .

Now using Lemma 2, we see that $\mathcal{C}_{\Gamma, \lambda}^T$ can be considered as the category of pairs of vector spaces E_p and E_q equipped with linear maps $R_p : E_p \rightarrow E_p$, $R_q : E_q \rightarrow E_q$, $E_a : E_p \rightarrow E_q$, $\nabla_a : E_q \rightarrow E_p$ satisfying

$$R_p - \lambda_p 1 = \nabla_a E_a \quad \text{and} \quad R_q - \lambda_q 1 = -E_a \nabla_a,$$

and with $\nabla_a E_a$ having eigenvalues in T .

On the other hand, ℓ_p and ℓ_q freely generate the fundamental groups of $X_1 \setminus \{p\}$ and $X_2 \setminus \{q\}$, so an object in $\text{Rep}_\sigma^S \pi(\Gamma)$ is given by vector spaces V_1, V_2 , elements $\rho_1(\ell_p) \in \text{GL}(V_1)$ and $\rho_2(\ell_q) \in \text{GL}(V_2)$ and linear maps $\rho_a : V_1 \rightarrow V_2$ and $\rho_a^* : V_2 \rightarrow V_1$ satisfying and

$$\sigma_p^{-1} \rho_1(\ell_p)^{-1} = 1 + \rho_a^* \rho_a \quad \text{and} \quad \sigma_q \rho_2(\ell_q) = 1 + \rho_a \rho_a^*$$

and such that the eigenvalues of $\rho_a \rho_a^*$ are in S .

The equivalence between these categories is given by the case $m = 2$ of Lemma 1, sending the object given by $E_p, E_q, R_p, R_q, E_a, \nabla_a$ to the object with $V_1 = E_p, V_2 = E_q, \rho_a = E_a$ and

$$\rho_a^* = \sum_{n=1}^{\infty} \frac{(2\pi\sqrt{-1})^n}{n!} \nabla_a (E_a \nabla_a)^{n-1}.$$

□

6. CYCLIC QUIVER EXPONENTIAL

Let Q_m be the cyclic quiver with vertices $1, \dots, m$ and arrows $a_1 : m \rightarrow 1$ and $a_i : i-1 \rightarrow i$ ($i = 2, \dots, m$). Given a subset $\Sigma \subseteq \mathbb{C}$ with $0 \in \Sigma$, we denote by $\text{Rep}^\Sigma Q_m$ the category of representations of Q_m in which the path $a_m \dots a_1$ (and hence, since $0 \in \Sigma$, any path of length m) is represented by an endomorphism with eigenvalues in Σ . Let T be a non-resonant set with $0 \in T$ and let $S = \{e^{2\pi\sqrt{-1}t} - 1 : t \in T\}$.

Lemma 1. *There is an equivalence $\text{Rep}^T Q_m \rightarrow \text{Rep}^S Q_m$ sending a representation of Q_m by vector spaces V_i and linear maps $a_i : V_{i-1} \rightarrow V_i$*

to the representation with the same vector spaces, and linear maps

$$a'_1 = \sum_{n=1}^{\infty} \frac{(2\pi\sqrt{-1})^n}{n!} a_1(a_n \dots a_1)^{n-1}$$

and $a'_i = a_i$ for $i \neq 1$.

Proof. The functor F given by this construction, and which acts as the identity on morphisms, defines an equivalence from the category of nilpotent representations to itself, as an inverse is given by the analogous logarithm functor, with

$$a''_1 = \frac{1}{2\pi\sqrt{-1}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} a_1(a_n \dots a_1)^{n-1}.$$

Given an endomorphism σ of a vector space V , let $B(\sigma)$ be the representation of Q_m in which all $V_i = V$, $a_1 = \sigma$ and $a_i = 1$ for $i \neq 1$. Recall that any indecomposable representation of Q_m is either nilpotent or isomorphic to $B(\sigma)$ for some indecomposable automorphism σ . Now if σ is given by a Jordan block $J_n(\lambda)$, then

$$F(B(\sigma)) = B(e^{2\pi\sqrt{-1}}\sigma - 1) \cong B(J_n(e^{2\pi\sqrt{-1}}\lambda - 1)).$$

It follows that F induces a surjective map onto the isomorphism classes of $\text{Rep}^S Q_m$. It is a faithful functor, and also full since the non-resonance of T ensures that

$$\dim \text{Hom}(F(X), F(Y)) = \dim \text{Hom}(X, Y)$$

for any X, Y nilpotent or of the form $B(\sigma)$. \square

7. LOGARITHMIC CONNECTIONS FOR A DISK

The result in this section is well known, but we state it carefully as the literature can be confusing.

Let X an open unit disk in \mathbb{C} with centre p , let $D = \{p\}$, and let Σ be a non-resonant subset of \mathbb{C} . Let \mathcal{C}^Σ be the category of pairs (E, ∇) consisting of a vector bundle E on X and a logarithmic connection $\nabla : E \rightarrow \Omega_X^1(\log D) \otimes E$, such that the eigenvalues of $\text{Res}_p \nabla$ are in Σ . A representation of the polynomial ring $\mathbb{C}[x]$ can be thought of as a pair (V, R) where V is a vector space and $R \in \text{End}(V)$. We denote by $\text{Rep}^\Sigma \mathbb{C}[x]$ be the category of representations such that R has eigenvalues in Σ . Given (V, R) , one obtains a trivial vector bundle $E = \mathcal{O} \otimes_{\mathbb{C}} V$ and a logarithmic connection $\nabla_R : E \rightarrow \Omega_X^1(\log D) \otimes E$ defined by

$$\nabla_R(f) = df + \frac{R}{z-p} \cdot f dz$$

for f a section of E . It has residue $\text{Res}_p \nabla_R = R$.

Lemma 2. *The functor $F(V, R) = (\mathcal{O} \otimes V, \nabla_R)$ induces an equivalence $\text{Rep}^\Sigma \mathbb{C}[x] \rightarrow \mathcal{C}^\Sigma$.*

Proof. By [3, Theorem 6.1], monodromy gives an equivalence M from \mathcal{C}^Σ to $\text{Rep}^H \pi_1(X \setminus D)$, the category of representations of the fundamental group of $X \setminus D$ in which the loop around p is given by a matrix with eigenvalues in $H = \{e^{-2\pi\sqrt{-1}\sigma} : \sigma \in \Sigma\}$.

The composition $MF : \text{Rep}^\Sigma \mathbb{C}[x] \rightarrow \text{Rep}^H \pi_1(X \setminus D)$ sends (V, R) to the representation with vector space V and in which the loop around p is given by $e^{-2\pi\sqrt{-1}R}$, so by properties of the matrix exponential (as in the case $m = 1$ of Lemma 1) it is an equivalence. It follows that F is an equivalence. \square

8. MULTIPLICATIVE PREPROJECTIVE ALGEBRAS

In case the Riemann surface quiver Γ is compact, we show that the target of the monodromy functor can be identified with a category of representations of a multiplicative preprojective algebra [5].

Let Q be the quiver obtained from the component quiver $[\Gamma]$ of Γ by adjoining g_i loops τ_i^j ($1 \leq j \leq g_i$) at each vertex i , where g_i is the genus of X_i . Define $q = (q_i) \in (\mathbb{C}^\times)^I$ by $q_i = e^{2\pi\sqrt{-1}\lambda_i}$ for $i \in I$, and let $\Lambda^q(Q)$ be the corresponding multiplicative preprojective algebra. We denote by $\text{Rep } \Lambda^q(Q)'$ the category of representations of $\Lambda^q(Q)$ in which the linear maps representing the loops τ_i^j are invertible. It can also be defined as the category of representations of a universal localization of $\Lambda^q(Q)$.

Proposition 2. *$\text{Rep}_\sigma \pi(\Gamma)$ is equivalent to $\text{Rep } \Lambda^q(Q)'$.*

Proof. If a_1, \dots, a_r are the arrows with head in D_i and b_1, \dots, b_s are the arrows with tail in D_i , then $\pi(X_i \setminus D_i)$ can be generated by elements $u_1, \dots, u_{g_i}, v_1, \dots, v_{g_i}, \ell_1, \dots, \ell_{r+s}$, subject to the relation

$$u_1 v_1 u_1^{-1} v_1^{-1} \cdots u_{g_i} v_{g_i} u_{g_i}^{-1} v_{g_i}^{-1} \ell_1 \cdots \ell_{r+s} = 1$$

where ℓ_1, \dots, ℓ_r are loops in $X_i \setminus D_i$ around the heads of a_1, \dots, a_r and $\ell_{r+1}, \dots, \ell_{r+s}$ are loops around the tails of b_1, \dots, b_s .

A representation $\rho : \pi(X_i \setminus D_i) \rightarrow \text{GL}(V_i)$ is determined by automorphisms $\rho(u_j), \rho(v_j), \rho(\ell_j)$ of V_i satisfying the same relation. Rewriting $\rho(u_j)$ and $\rho(v_j)$ in terms of endomorphisms e_j and e_j^* of V_i via

$$\rho(u_j) = e_j \quad \text{and} \quad \rho(v_j) = e_j^{-1} + e_j^*,$$

and using the equations

$$\sigma_p^{-1}\rho_i(\ell_p)^{-1} = 1_{V_i} + \rho_a^*\rho_a \quad \text{and} \quad \sigma_q\rho_j(\ell_q) = 1_{V_j} + \rho_a\rho_a^*,$$

the relation becomes

$$(1 + e_1 e_1^*)(1 + e_1^* e_1)^{-1} \cdots (1 + e_{g_i} e_{g_i}^*)(1 + e_{g_i}^* e_{g_i})^{-1} \cdot (1 + \rho_{a_1} \rho_{a_1}^*) \cdots (1 + \rho_{a_r} \rho_{a_r}^*)(1 + \rho_{b_1}^* \rho_{b_1})^{-1} \cdots (1 + \rho_{b_s}^* \rho_{b_s})^{-1} = q_i 1$$

This is the relation at vertex i for the multiplicative preprojective algebra $\Lambda^q(Q)$, where e_j and e_j^* represent τ_i^j and τ_i^{j*} . Observe that the requirement for the multiplicative preprojective algebra that all terms in this product be invertible, coupled with the requirement that the τ_i^j be represented by invertible linear maps, corresponds to the fact that the generators of the fundamental group are given in the representation ρ by invertible linear maps. \square

9. COROLLARIES

We obtain the following answer to a question of Shaw [9, §5.2]. Note that if one works not over \mathbb{C} , but over a base field of characteristic 2, then the assertion is false, see [9, Lemma 5.5.1]

Corollary 1. *If Q is a quiver of Dynkin type ADE, then $\Lambda^1(Q)$ is isomorphic to the usual preprojective algebra $\Pi(Q)$.*

Proof. Let Γ be a Riemann surface quiver of \mathbb{P}^1 -type whose component quiver is Q . The Monodromy Theorem gives an equivalence $\mathcal{C}_{\Gamma,0}^T \rightarrow \text{Rep } \Lambda^q(Q)$ for a suitable transversal T .

By [6, Lemma 5.1], the simple representations for $\Lambda^1(Q)$ are 1-dimensional at a vertex. Thus they come from vector bundle representations and 0-connection systems which consist of a single line bundle equipped with a holomorphic connection. Moreover the line bundles have degree 0 (for example by the Lifting Theorem), so, since Γ is of \mathbb{P}^1 -type, they are trivial.

Now any representation of $\Lambda^1(Q)$ is an iterated extension of simple representations, so any object of $\mathcal{C}_{\Gamma,0}^T$ is an iterated extension of trivial line bundle systems. Since the trivial line bundle on \mathbb{P}^1 has no self-extensions, it follows that any object of $\mathcal{C}_{\Gamma,0}^T$ is vb-trivial. Thus by Proposition 1, the objects of $\mathcal{C}_{\Gamma,0}^T$ can be considered as representations of $\Pi(Q)$. Now as Q is Dynkin, $\Pi(Q)$ is finite-dimensional and any representation is nilpotent. It follows that $\mathcal{C}_{\Gamma,0}^T$ is equivalent to $\text{Rep } \Pi(Q)$.

Thus the Monodromy Theorem gives an equivalence $\text{Rep } \Pi(Q) \rightarrow \text{Rep } \Lambda^1(Q)$. Here we have only considered finite-dimensional representations, but $\Lambda^1(Q)$ is finite-dimensional by [9, 3.1.1], so this equivalence is a Morita equivalence. Finally, as the simple representations are 1-dimensional for both algebras, the algebras are isomorphic. \square

This gives the following simple case of Hilbert's 21st problem (which is presumably already known).

Corollary 2. *Given n_1, n_2, n_3 with $1/n_1 + 1/n_2 + 1/n_3 > 1$ and $D = \{a_1, a_2, a_3\} \subset \mathbb{P}^1$, any representation $\rho : \pi_1(\mathbb{P}^1 \setminus D) \rightarrow \text{GL}_n(\mathbb{C})$ sending loops around the a_i to unipotent matrices ρ_i with $(\rho_i - 1)^{n_i} = 0$ arises as the monodromy of a logarithmic connection on a trivial vector bundle (so a Fuchsian system).*

Proof. We may assume that $\rho_1 \rho_2 \rho_3 = 1$. The representation corresponds, as in [6, §8], to a representation of $\Lambda^1(Q)$ for some star-shaped quiver Q . The condition on n_i ensures that Q is of Dynkin type. By the previous result it arises from some representation of $\Pi(Q)$, which can be interpreted as a vb-trivial object in $\mathcal{C}_{\Gamma,0}^T$ as in the previous proof. The connection on the trivial vector bundle at the central vertex of Q is the required Fuchsian system. \square

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